The Dispersion of a Light Pulse by a Prism.

By R. A. HOUSTOUN, M.A., Ph.D., D.Sc., Lecturer on Physical Optics in the University of Glasgow.

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This investigation is a continuation of a former one in which an expression was derived for a light pulse with an energy distribution given by Wien's law.* The first three paragraphs are supplementary to the former paper; the rest of the investigation deals with the passage of the same pulse through a prism and its separation into the different colours in the focal plane of a telescope. The general principles according to which this must take place are, of course, known, but here the actual disturbance at every point in the focal plane is given for the first time as a definite function of the time and as a result it is possible to state how many waves there are in the trains, which the single initial pulse gives rise to in the various parts of the spectrum.

§ 1. My general expression for the initial form of a light pulse was

$$\frac{\cos\left(n+\frac{1}{2}\right)\theta}{(x^2+h^2)^{(2n+1)/4}},$$

where $\tan\theta = x/h$. I did not notice until after the former paper was communicated, that this expression is

$$\frac{1}{\Gamma\left(n+\frac{1}{2}\right)}\int_0^\infty e^{-h\alpha}\cos\alpha x\,\alpha^{n-\frac{1}{2}}d\alpha.$$

The pulse is thus equivalent to the superposition of an infinite series of cosine waves of gradually increasing wave-length, the parameter of the wave being connected with the wave-length in air by $\alpha = 2\pi/\lambda$ and the amplitude being given by $e^{-h\alpha}\alpha^{n-\frac{1}{2}}$. According to the former paper the velocity of such a wave is $1/\sqrt{(\alpha c)}$. Hence after time t the disturbance becomes

$$\frac{1}{\Gamma\left(n+\frac{1}{2}\right)}\int_0^\infty e^{-h\alpha}\cos\alpha\left(x-\frac{t}{\sqrt{(\alpha c)}}\right)\alpha^{n-\frac{1}{2}}d\alpha.$$

But, when t^2/c was large, the disturbance was shown to be proportional to

$$\frac{t^{2n}}{x^{2n+\frac{1}{2}}}e^{-t^2h/4cx^2}\cos\left(t^2/4cx\right).$$

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* "The Mathematical Representation of a Light Pulse," 'Roy. Soc. Proc.,' A, vol. 89, p. 399 (1913).

Consequently on putting in the constants which were neglected formerly for the sake of simplicity we arrive by physical reasoning at the new integral,

$$\left(\frac{t^2}{c}\text{ large}\right) \qquad \int_0^\infty e^{-h\alpha} \cos \alpha \left(x - \frac{t}{\sqrt{(\alpha c)}}\right) \alpha^{n-\frac{1}{2}} d\alpha = \frac{\sqrt{\pi} t^{2n}}{4^n c^n x^{2n+\frac{1}{2}}} e^{-t^2 h/4cx^2} . \cos (t^2/4cx).$$

§ 2. Planck's law for black body radiation is

$$\frac{c_1}{\lambda^5} \cdot \frac{1}{e^{c_2/\lambda \mathrm{T}} - 1} = \frac{c_1}{\lambda^5} \cdot \frac{e^{-c_2/\lambda \mathrm{T}}}{1 - e^{-c_2/\lambda \mathrm{T}}}.$$

Except for high temperatures and in the far infra-red, $e^{-c_2/\lambda \mathrm{T}}$ is small in comparison with 1. Hence the denominator may be expanded as a series and the law becomes

$$\frac{c_1}{\lambda^5} [e^{-c_2/\lambda \mathrm{T}} + e^{-2c_2/\lambda \mathrm{T}} + e^{-3c_2/\lambda \mathrm{T}} + \ldots],$$

each term being of the form of Wien's law but with the constants multiples of the Wien law constant. It follows, therefore, that the Planck distribution must be given by the initial disturbance formed by superposing the pulses which would give these separate Wien distributions. That is, in the notation of my previous paper, the pulse initially represented by the series

$$\frac{\cos \frac{5}{2} \tan^{-1} (x/h)}{(x^2 + h^2)^{5/4}} + \frac{\cos \frac{5}{2} \tan^{-1} (x/2h)}{(x^2 + \{2h\}^2)^{5/4}} + \frac{\cos \frac{5}{2} \tan^{-1} (x/3h)}{(x^2 + \{3h\}^2)^{5/4}} + \ldots$$

gives a distribution of energy obeying Planck's law. This was first pointed out to me by Dr. Green.

In fig. 1 the dotted curve represents the first term of the series and the full curve the sum of the first ten terms of the series both drawn for $h = 1$. The addition of the remaining terms of the series would at the most raise the full curve 0·02 all over. There is thus not much difference in the shape of the pulses giving the Wien and Planck distributions.

As shown in the former paper h varies inversely as the absolute temperature. If the absolute temperature is doubled, h is halved and it is seen from the form of the curve that the shape of the pulse at the new temperature can be obtained from the first shape by multiplying the ordinates throughout by $2^{5/2}$, *i.e.* by 5·56, and at the same time moving each of them half way into the centre. As the temperature increases the pulse thus becomes narrower and sharper. The value $h = 1$ corresponds to an absolute temperature of $\mathrm{T} = c_2/(4\pi) = 1{\cdot}436/(4\pi) = 0{\cdot}114^\circ$.

The curves representing the successive terms of the series can be obtained from the curve representing the first term by multiplying the ordinates by 0·177, 0·064, 0·031, 0·018, etc., and at the same time moving them out twice, thrice, four times, five times, etc., their distance from the centre. The full curve in fig. 1 showing the first 10 terms of the series was obtained in this way by constructing the different terms from the first graphically and then summing then.

§ 3. The expressions derived in the previous paper represented plane light pulses moving in the direction of the x axis. A little consideration shows that they can easily be adapted to the three-dimensional case of symmetry about a point.

Consider the equation

$$\frac{\partial^2}{\partial r^2}(rS)+c^2\frac{\partial^4}{\partial t^4}(rS)=0.$$

It is solved by $rS=\sin\frac{2\pi}{\tau}\left(t-\frac{r}{v}\right)$, where $v=\pm\tau/(2\pi c)$. S consequently represents a wave diverging from the origin in a medium in which the velocity is directly proportional to the period. By analogy with equation (1) of the former paper

$$rS=\frac{\partial^n}{\partial(ir)^n}\cdot\frac{1}{\sqrt{(h+ir)}}e^{-t^2/4c(h+ir)}$$

is a solution of the above equation. Consequently our former solutions may

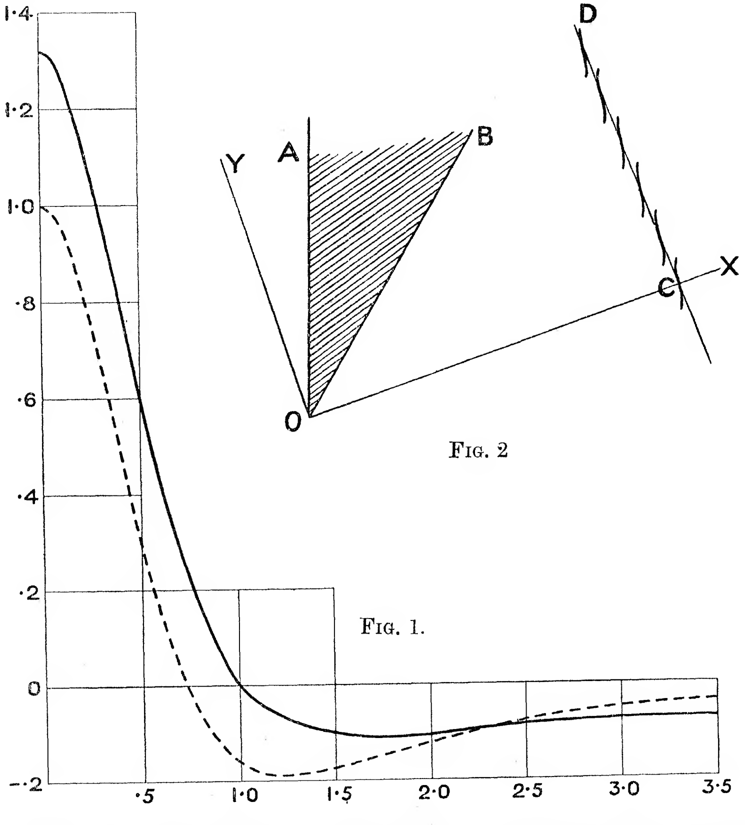

Fig. 1.

Fig. 2

be adapted for three-dimensional motion by the simple artifice of writing r for x and multiplying by $1/r$. For example, the initial shape of the disturbance with the Planck distribution of energy is

$$\sum_{n=1}^{n=\infty} \frac{\cos\frac{5}{2}\tan^{-1}(r/nh)}{r\,(r^2+\{nh\}^2)^{5/4}}.$$

§ 4. Let a plane light pulse with the Wien energy distribution be incident normally on the face OA of the prism AOB (fig. 2). Then before incidence it may be represented by

$$\int_0^\infty e^{-h\alpha}\cos\alpha\,(l-vt)\,\alpha^{3/2}d\alpha,$$

where l is distance from OA measured positive from left to right and v is the velocity of light in air. The prism is situated in air and has a refracting angle of 30°. In this section we shall find the shape of the disturbance immediately after passing through the prism. For this purpose we shall ignore the loss by reflection at the faces OA and OB, and shall also assume the prism to be unlimited in the direction OA.

The treatment depends on the principle of the superposition of small vibrations. Let us suppose we have a series of infinitely long trains of cosine waves incident on the prism and that the amplitude of each train is given in terms of its parameter by $e^{-h\alpha}\alpha^{\frac{3}{2}}$. The direction of each train after its emergence from the prism is known, since the index of refraction of the latter is supposed given, and the phase of the emergent wave is also known, since it coincides with the phase of the incident wave at O. Then, since the impulse is equivalent to the superposed cosine trains before incidence, it must be equivalent to them after emergence. It must be observed that in no physical sense are the cosine trains contained in the impulse.

Let OX be the direction of emergence of the cosine train with the greatest amplitude. Denote the parameter of the latter by α_0; then it is easy to show that $\alpha_0 = 3/2\,h$. This train is represented after emergence by

$$e^{-h\alpha_0}\cos\alpha_0\,(x-vt)\,\alpha_0{}^{3/2}.$$

The other trains make small angles with OX after emergence and hence must be represented by

$$e^{-h\alpha}\cos\alpha\,(x\cos\theta+y\sin\theta-vt)\,\alpha^{3/2}, \qquad (1)$$

where θ is a function of α. Let us assume that $\sin\theta = k\,(1/\alpha_0-1/\alpha)$, where k is a constant; this makes the deviation vary with the wave-length in the right direction and gives a function capable of integration. Also assume θ so small that $\cos\theta$ may be put $= 1$. Then after emergence from the prism the total disturbance is given by

$$\int_0^\infty e^{-h\alpha}\cos\alpha\left(x+k\left\{\frac{1}{\alpha_0}-\frac{1}{\alpha}\right\}y-vt\right)\alpha^{3/2}\,d\alpha$$

$$=\int_0^\infty e^{-h\alpha}\cos\left[\alpha\,\{x+(k/\alpha_0)\,y-vt\}\,-ky\right]\alpha^{3/2}\,d\alpha$$

$$=\int_0^\infty e^{-h\alpha}\left[\cos\alpha\,\{x+(k/\alpha_0)\,y-vt\}\cos ky+\sin\alpha\,\{x+(k/\alpha_0)\,y-vt\}\sin ky\right]\alpha^{3/2}\,d\alpha$$

$$=\frac{\Gamma(\frac{5}{2})\cos\frac{5}{2}\tan^{-1}\left[\{x+(k/\alpha_0)\,y-vt\}/h\right]\cos ky}{[h^2+\{x+(k/\alpha_0)\,y-vt\}^2]^{5/4}}$$
$$+\frac{\Gamma(\frac{5}{2})\sin\frac{5}{2}\tan^{-1}\left[\{x+(k/\alpha_0)\,y-vt\}/h\right]\sin ky}{[h^2+\{x+(k/\alpha_0)\,y-vt\}^2]^{5/4}}$$

$$=\frac{\Gamma(\frac{5}{2})\cos\left[\frac{5}{2}\tan^{-1}(1/h)\,\{x+(k/\alpha_0)\,y-vt\}\,-ky\right]}{[h^2+\{x+(k/\alpha_0)\,y-vt\}^2]^{5/4}}.$$

It is evident that after passing through the prism the disturbance is still a pulse and of much the same thickness as before. Its characteristics are best studied by giving the constants definite numerical values.

Suppose that the temperature of the source is 6000° absolute, the commonly accepted value for the sun. Then

$$h=c_2/4\pi\mathrm{T}=1{\cdot}436/24{,}000\pi=1{\cdot}904\,.\,10^{-5}$$

and
$$\alpha_0=3/(2\times1{\cdot}904\times10^{-5})=7{\cdot}87\,.\,10^4.$$

The wave-length corresponding to α_0 is $7{\cdot}97\,.\,10^{-5}$ cm., a value for the extreme red end of the spectrum.

If the prism were made of light barium crown glass, the index of refraction for α_0 would be 1·512, for $\alpha=1{\cdot}066\,.\,10^5$ (D lines) it would be 1·517, for $\alpha=1{\cdot}292\,.\,10^5$ (blue hydrogen line) it would be 1·523, and for $\alpha=1{\cdot}552\,.\,10^5$ (violet mercury line) it would be 1·5318. The deviations for these wave-lengths would then be respectively 19° 7′, 19° 20′, 19° 36′, and 20° 0′. The differences of the last three figures and the first are 13′, 29′, and 53′. It is impossible to fix our constant k so as to make the deviations accord exactly with those of any known glass, but if we take $1{\cdot}8\,.\,10^3$ as its value, the excess of the deviations of the last three wave-lengths over the first would be 20′, 31′, and 38′. We shall take this value for k; thus the dispersive power of the material of our prism is somewhat similar to that of crown glass.

Let us now examine the distribution of the disturbance for $t=10^{-10}$ sec., *i.e.* the time taken by a cosine train to move out 3 cm. from the origin. Write $x-vt=\xi$, *i.e.* $\xi=x-3$ cm. Then, if the constant factor be ignored, the disturbance is the product of

$$\frac{1}{[h^2+\{\xi+(k/\alpha_0)\,y\}^2]^{5/4}}\quad\text{and}\quad\cos\left[\tfrac{5}{2}\tan^{-1}(1/h)\,\{\xi+(k/\alpha_0)\,y\}-ky\right].\qquad(2)$$

The first factor is a maximum along the straight line $\xi+(k/\alpha_0)y = 0$, *i.e.* $\xi+0{\cdot}0229\,y = 0$. This straight line is represented by CD in fig. 2. The first factor diminishes to zero on both sides of the maximum and has reached 1/100 of its maximum value at a distance of $\pm\,1{\cdot}19\,.\,10^{-4}$ cm. from CD. We shall consider the variation of the second factor only within this distance from CD.

As ξ varies from $-\infty$ to $+\infty$, the second factor varies from $\cos\{-(5\pi/4)-ky\}$ to $\cos\{+(5\pi/4)-ky\}$. It cannot thus have more than two maxima for one value of y. The locus of its maxima is given by

$$\tfrac{5}{2}\tan^{-1}(1/h)\,\{\xi+(k/\alpha_0)\,y\}-ky = 2n\pi,$$

i.e.,
$$(1/h)\,\{\xi+(k/\alpha_0)\,y\} = \tan\tfrac{2}{5}(2n\pi+ky), \tag{3}$$

which obviously represents a family of curves, one for each value of n. These curves give the position of the crests.

In equation (3) keep $\xi+(k/\alpha_0)\,y = 0$ and gradually increase y, *i.e.*, move along the straight line CD. Whenever $ky = -\,2n\pi$, the equation is satisfied, that is, whenever

$$y = -2n\pi/1{\cdot}8\,.\,10^3.$$

This equation therefore gives the points in which the curves cut CD. They are equidistant and about $3{\cdot}48\,.\,10^{-3}$ cm. apart.

From (3) we have

$$\frac{d\xi}{dy}+\frac{k}{\alpha_0} = \frac{2hk}{5}\sec^2\tfrac{2}{5}(2n\pi+ky). \tag{4}$$

If we write p to denote the perpendicular distance of the point (ξ, y) from the line CD, then, by (3),

$$\sec^2\tfrac{2}{5}(2n\pi+ky) = 1+\tan^2\tfrac{2}{5}(2n\pi+ky) = 1+\frac{p^2}{h^2}.$$

On substituting this in (4) we have

$$\frac{d\xi}{dy} = -\frac{k}{\alpha_0}+\frac{2hk}{5}\left(1+\frac{p^2}{h^2}\right).$$

All the curves are therefore parallel for a given value of p.

When the first factor has diminished to 1/100 of its maximum value $p^2/h^2 = 39$. On substituting this and the values for h, k and α_0 we find

$$\frac{d\xi}{dy} = -0{\cdot}0229+0{\cdot}0137\times 40 = 0{\cdot}53.$$

For CD itself $d\xi/dy$ has the value $-0{\cdot}0229$.

From the above results an idea of the position of the crests may be obtained and they are represented in fig. 2, very much out of scale, by the short curves crossing CD.

The equation of the crest through C is obtained from (3) by putting $n = 0$ and may be written as
$$p = h \tan \tfrac{2}{5} ky.$$
I have drawn it to scale in fig. 3. CD denotes the same line as in fig. 2 and the lines through G and H parallel to CD are those for which the first factor has 1/100 its maximum value. The ends of the neighbouring crests come into the figure. As has been pointed out, all the crests are equally spaced and have the same shape.

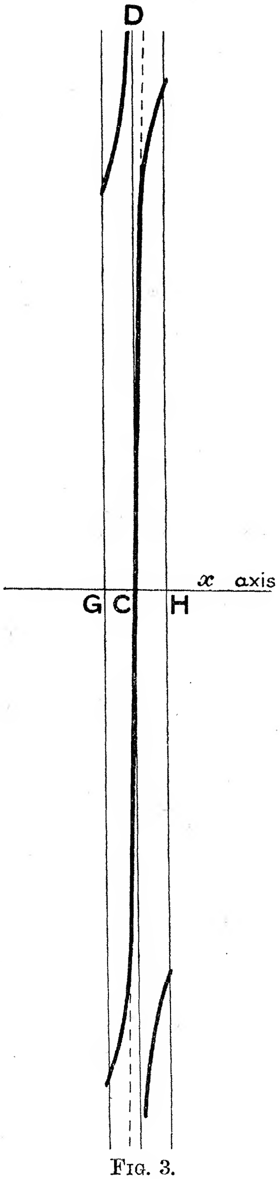

Fig. 3.

It is obvious from fig. 2 that the shape of the disturbance is analogous to the ship waves that would be produced by a pressure point moving along OB. This analogy has been used by Dr. Green.*

An important assumption has been passed over in silence, namely, that $\cos\theta$ may be put $= 1$ in (1). In the violet when $\theta = 38'$, $\cos\ \theta = 1 - 3{\cdot}1\ 10^{-5}$, x is 3 and α is $1{\cdot}552 \,.\, 10^5$. Hence the error introduced in the phase is the large one of 14·4 radians. It diminishes towards the red. It seems, however, from the applications of the principle of stationary phase, that only interference between waves of nearly equal wave-length is important and the relative phase difference of such waves is not affected appreciably by the approximation. In any case the results are rigorous for the neighbourhood of the x axis. If the accurate value is substituted for $\cos\ \theta$, the expression will not integrate.

§ 5. In the previous section the nature of the light was considered immediately after emerging from the prism, before the colours have time to separate out, in the region where, if a screen is interposed, a white rectangular patch is obtained tinged on the one side with red and on the other with violet. We shall now investigate the disturbance in the focal plane of the spectroscope telescope.

Hitherto diffraction has been ignored. It is now fundamental. Let the pulse be incident normally on the face OA as before, but let there be a screen (fig. 4) on the other face restricting the length OB that can be used. Let b be the greatest thickness of the prism used. After emergence from the prism the light is received by a lens which has the property of making a parallel beam converge to a mathematical point. The diameter of the lens is so large that it receives all the rays that pass through the aperture OB; the latter causes the diffraction, not the rim of the lens. Reflection and absorption losses are ignored both in the prism and the lens.

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* "Illustration of the *Modus operandi* of the Prism," 'Roy. Soc. Edin. Proc.,' vol. 31, p. 290 (1911).

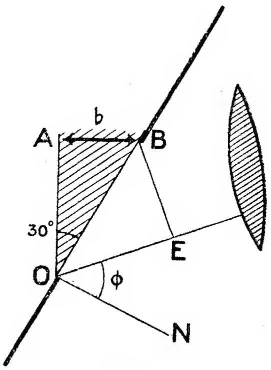

Fig. 4.

Each point on the face OB must be considered as sending rays of all possible wave-lengths in every direction. Take a definite direction OE, which is specified by the angle ϕ which it makes with ON, the normal to the face OB. To this direction there corresponds a point P in the focal plane of the lens. Let the optical distance of P from O, measured along the ray, be s.

Let us consider the disturbance at P due to rays of parameter α. We have to sum an infinite number of rays of equal intensity and period but uniformly increasing phase. Let $2p$ be the difference of phase in radian measure between the first and last of them. Then, as is well known, the amplitude of the resultant varies as $(\sin p)/p$.

If BE is drawn perpendicular to OE and μ denotes the index of refraction of the prism, OE = OB $\sin\phi = 2b\sin\phi$, and consequently

$$2p = \frac{2\pi}{\lambda}(\mu b - 2b\sin\phi) = \alpha b\,(\mu - 2\sin\phi).$$

The phase of the resultant is the phase of the mean ray. Consequently the disturbance at P is given by

$$\frac{\sin p}{p}\cos\{\alpha(s-vt)+p\}.$$

To get the whole disturbance at P we have to multiply the above expression by $e^{-h\alpha}\,\alpha^{3/2}$ in order to take account of the shape of the initial pulse, and then integrate with respect to α. The result is

$$\int_0^\infty e^{-h\alpha}\frac{\sin p}{p}\cos\{\alpha(s-vt)+p\}\,\alpha^{3/2}d\alpha\,;$$

p is of course a function of α. For the expression to be integrable we are restricted to one function, $g-f\alpha$. We shall assume this value, holding over the question of its justification. Meantime it should be noted that f and g are constants of the prism and independent of the shape of the incident pulse.

As we move through the spectrum we know from its physical nature that $(\sin p)/p$ remains zero until we come to one particular point, then rises suddenly to a peak and immediately decreases as suddenly to zero again. The other two factors of the cosine in the integrand vary so slowly within this region that they may be taken outside the integral sign. We have left then

$$\int_0^\infty \frac{\sin(g-f\alpha)}{(g-f\alpha)} \cos\{\alpha(s-vt)+g-f\alpha\}\,d\alpha.$$

Write $\alpha-g/f=\beta$ and $g/f=\alpha_1$. Then $\beta=\alpha-\alpha_1$ and the integral becomes

$$\int_{-\alpha_1}^{\infty} \frac{\sin f\beta}{f\beta} \cos\{\beta(s-vt-f)+\alpha_1(s-vt)\}\,d\beta$$

$$=\int_{-\alpha_1}^{\infty} \frac{\sin\{\beta(s-vt)+\alpha_1(s-vt)\}-\sin\{\beta(s-vt-2f)+\alpha_1(s-vt)\}}{2f\beta}\,d\beta$$

$$=\int_{-\alpha_1}^{\infty} \frac{\sin\beta(s-vt)\cos\alpha_1(s-vt)+\sin\alpha_1(s-vt)\cos\beta(s-vt)}{2f\beta}-\ldots\ldots\ldots\ldots\,d\beta$$

$$=\cos\alpha_1(s-vt)\int_{-\alpha_1}^{\infty} \frac{\sin\beta(s-vt)-\sin\beta(s-vt-2f)}{2f\beta}\,d\beta$$

$$+\sin\alpha_1(s-vt)\int_{-\alpha_1}^{\infty} \frac{\cos\beta(s-vt)-\cos\beta(s-vt-2f)}{2f\beta}\,d\beta$$

$$=\frac{1}{2f}\cos\alpha_1(s-vt)\left[\int_{-\alpha_1(s-vt)}^{\infty(s-vt)} \frac{\sin y}{y}\,dy-\int_{-\alpha_1(s-vt-2f)}^{\infty(s-vt-2f)} \frac{\sin y}{y}\,dy\right]$$

$$+\frac{1}{2f}\sin\alpha_1(s-vt)\left[\int_{-\alpha_1(s-vt)}^{\infty(s-vt)} \frac{\cos y}{y}\,dy-\int_{-\alpha_1(s-vt-2f)}^{\infty(s-vt-2f)} \frac{\cos y}{y}\,dy\right]. \qquad (5)$$

The integrals in the brackets are known as the sine integral and cosine integral, and their values have been calculated for all values of y by J. W. L. Glaisher.*

The first bracket can be re-written

$$\int_{-\alpha_1(s-vt)}^{-\alpha_1(s-vt-2f)} \frac{\sin y}{y}\,dy+\int_{\infty(s-vt-2f)}^{\infty(s-vt)} \frac{\sin y}{y}\,dy.$$

Let us consider the second term first. It will be shown farther down that $2f$ is positive and much smaller than s. When $t=0$ both limits are $+\infty$; when $t=(s-2f)/v$ the lower limit changes to $-\infty$, and when $t=s/v$ the upper limit changes to $-\infty$. The value of the integral between the limits $\pm\infty$ is, of course, π. Hence from $t=(s-2f)/v$ to $t=s/v$ the integral is π; before and after this it equals zero.

Next consider the other term. When $t=0$ it is zero. As t approaches $(s-2f)/v$ it suddenly approaches $\frac{1}{2}\pi$ then as the upper limit changes sign and becomes positive, it increases to π. It will be shown farther down that the upper limit practically reaches $+\infty$ long before the lower limit changes sign. As the lower limit becomes zero and then positive the integral decreases again to zero. The changes are the same as for the second term, only they take place continuously.

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* 'Phil. Trans.,' vol. 160, pp. 367–387 (1870).

The second bracket can be written

$$\int_{-\alpha_1(s-vt)}^{-\alpha_1(s-vt-2f)} \frac{\cos y}{y} dy + \int_{\infty(s-vt-2f)}^{\infty(s-vt)} \frac{\cos y}{y} dy,$$

but this integral, though very great near the origin, diminishes rapidly to a small quantity, is zero at infinity, and so can be neglected in comparison with the other one.

Our final result, therefore, for the disturbance at P, when the term left outside the integral sign is included, is

$$\frac{\pi}{f} e^{-h\alpha_1}\alpha_1^{3/2} \cos \alpha_1 (s-vt) \tag{6}$$

between the times $t = (s-2f)/v$ and $t = s/v$. Outside these limits there is no disturbance.

We had $$p = \alpha b\left(\frac{\mu}{2} - \sin\phi\right) = g - f\alpha,$$

hence $$\frac{\mu}{2} = \sin\phi - \frac{f}{b} + \frac{g}{b\alpha}. \tag{7}$$

This gives $$\frac{d\mu}{d\lambda} = \frac{g}{\pi b}.$$

The dispersion is therefore uniform throughout the spectrum. Unfortunately it runs the wrong way; μ increases with λ. Let us suppose that $b = 2$ cm. and that $d\mu/d\lambda$ is 500, which is about the value for crown glass in the neighbourhood of the D lines. Then $g = 1000\pi$. If α_1 has the value for the D lines, $1{\cdot}066 \,.\, 10^5$, $f = g/\alpha_1 = 0{\cdot}0293$, and $2f\alpha_1 = 6{\cdot}28 \,.\, 10^3$. As the sine integral for $y = 75$ differs only by 1 per cent. from its value for infinity, the assumption made above is justified. It will be observed that it is not necessary to fix the value of μ, though from equation (7) it cannot be greater than 2 for the wave-length in question.

The number of complete wave-lengths in the train is $2f/\lambda = \alpha_1 f/\pi = g/\pi = b\frac{d\mu}{d\lambda}$, Rayleigh's expression for the resolving power. Its numerical value in this case is 1000.

It should be noted that the expression giving p in terms of α is always of the same form. For let ϕ have the value ϕ'. Then

$$p = \alpha b\left(\frac{\mu}{2} - \sin\phi'\right),$$

which gives, on substituting from (7) for $\mu/2$,

$$p = \alpha b\left(\sin\phi - \frac{f}{b} + \frac{g}{b\alpha} - \sin\phi'\right)$$

$$= g - \alpha\,(f + b\,\{\sin\phi' - \sin\phi\}),$$

which can be written $g - \alpha f'$. The value of g thus remains the same, but the value of f changes with ϕ.

We have
$$f' = f + b\,\{\sin\phi' - \sin\phi\},$$
which, on proceeding to the limit, gives

$$df = b\cos\phi\,d\phi.$$

But $f = g/\alpha$, hence $df = -(g/\alpha^2)\,d\alpha$, and consequently

$$g\,d\alpha = -\alpha^2 b\cos\phi\,d\phi. \tag{8}$$

§ 6. Let us now verify the analysis of the preceding section by integrating the energy in the different parts of the spectrum and comparing the result with the energy in the initial pulse.

The latter can be represented by

$$\int_{-\infty}^{+\infty} dx \left(\int_0^{\infty} e^{-h\alpha}\cos\alpha x\,\alpha^{3/2}d\alpha\right)^2 = \int_{-\infty}^{+\infty}\left\{\frac{\Gamma\left(\frac{5}{2}\right)\cos\frac{5}{2}\tan^{-1}(x/h)}{(x^2+h^2)^{5/4}}\right\}^2 dx,$$

which, by the preceding paper, is equal to

$$\frac{2}{h^4}\left\{\Gamma\left(\tfrac{5}{2}\right)\right\}^2\int_0^{\pi/2}\cos^2\frac{5\theta}{2}\cos^3\theta\,d\theta.$$

Now
$$\int_0^{\pi/2}\cos^2\frac{5\theta}{2}\cos^3\theta\,d\theta = \tfrac{1}{8}\int_0^{\pi/2}(1+\cos 5\theta)(\cos 3\theta + 3\cos\theta)\,d\theta$$

$$= \tfrac{1}{8}\int_0^{\pi/2}\left(\cos 3\theta + 3\cos\theta + \tfrac{1}{2}\{\cos 8\theta + \cos 2\theta\} + \tfrac{3}{2}\{\cos 6\theta + \cos 4\theta\}\right)d\theta$$

$$= \tfrac{1}{8}\left(-\tfrac{1}{3}+3\right) = \tfrac{1}{3}$$

and
$$\Gamma\left(\tfrac{5}{2}\right) = \left(\tfrac{3}{2}\right)\cdot\frac{\sqrt{\pi}}{2};$$

hence the incident energy can be represented by

$$\frac{3\pi}{8h^4}. \tag{9}$$

Before proceeding to find the energy in the spectrum an additional factor must be added to (6). It was stated that the amplitude of the wave diffracted in the direction ϕ was proportional to $(\sin p)/p$. We are now concerned with absolute values, and must determine the constant of proportionality, which for the sake of simplicity was previously omitted.

If a harmonic wave is incident normally on the prism, on emerging it is diffracted in a number of different directions which are grouped symmetrically about the direction given by $p = 0$. The energy in the range comprised between ϕ and $\phi + d\phi$ is proportional to $\{(\sin p)/p\}^2 d\phi$. Now $p = \alpha b(\mu/2 - \sin\phi)$, hence since we are here confined to one wave-length $dp = \alpha b \cos\phi\, d\phi$. The energy in this range is therefore proportional to

$$\left(\frac{\sin p}{p}\right)^2 \cdot \frac{dp}{\alpha b \cos\phi}.$$

The energy in the incident wave has been taken implicitly as 1; hence, if we integrate the above expression over all values of p, divide the result into 1 and take the square root, we shall find the desired constant. The result is

$$\sqrt{\left(\frac{\alpha b \cos\phi}{\pi}\right)}.$$

The more accurate expression to be used instead of (6) is therefore, on omitting the suffixes,

$$\frac{\pi}{f}\sqrt{\left(\frac{\alpha b \cos\phi}{\pi}\right)}\, e^{-h_\alpha \alpha^{3/2}} \cos\alpha\,(s - vt). \tag{10}$$

To measure the energy in this wave take the mean square of the amplitude and multiply by the length of the train, $2f$. The result is

$$\frac{1}{2}\left(\frac{\pi\alpha b \cos\phi}{f^2}\, e^{-2h_\alpha \alpha^3}\right) 2f.$$

This as yet represents only the energy in a single direction and is of course infinitely small. To find the energy in the whole spectrum we must multiply by $d\phi$ and integrate over all possible values of ϕ. This is done by evaluating $d\phi$ in terms of $d\alpha$ by means of (8). The result is then

$$-\pi \int_0^\infty e^{-2h_\alpha \alpha^3}\, d\alpha = \frac{3\pi}{8h^4},$$

which agrees perfectly with (9).

§ 7. To show how the amplitude varies at the beginning of the train I have drawn up the following table. The first, fourth, and seventh columns

−6·28	+0·14	−0·030	0·000	+4·69	−∞	+0·628	+5·30	−0·014
−5·65	+0·12	−0·110				+1·25	+5·84	−0·43
−5·02	+0·05	−0·188				+1·89	+6·24	−0·44
−4·39	−0·10	−0·187				+2·52	+6·49	−0·28
−3·76	−0·23	−0·095				+3·13	+6·54	+0·076
−3·13	−0·28	−0·076				+3·76	+6·49	+0·095
−2·52	−0·21	+0·28				+4·39	+6·36	+0·187
−1·89	+0·02	+0·44				+5·02	+6·21	+0·188
−1·25	+0·42	+0·43				+5·65	+6·14	+0·110
−0·628	+0·96	+0·014				+6·28	+6·12	+0·030

give the phase in radians measured from the time $t = (s - 2f)/v$; it increases in steps of $\pi/5$. The figures with the minus sign denote time before the arrival of the wave proper. The second, fifth, and eighth columns give the value of the first square bracket in (5), and the third, sixth, and ninth columns the value of the second square bracket.

The table shows that 97 per cent. of the growth of the amplitude takes place within two wave-lengths and, as mentioned above, there are 1000 wave-lengths in the train. It thus starts very suddenly. The infinite amplitude in the second column does not seem to influence the total energy appreciably as it was neglected in § 6.

At the end of the train the changes are gone through in the reverse order, and in addition the sign of the infinite amplitude is changed.

§ 8. It is obvious that the results in § 5 do not depend on the shape of the prism or on the fact that the pulse is incident perpendicularly. Also, although derived for one particular law of dispersion, they hold for any law of dispersion. The $(\sin p)/p$ term varies so rapidly with μ that $d\mu/d\lambda$ can be taken as constant within the range of integration. Now f was shown to equal $(b\lambda/2)\, d\mu/d\lambda$. Hence the results may be made independent of our particular law of dispersion by the substitution of this value for f.

§ 9. Hitherto the incident disturbance has been a pulse. Let us now suppose it to be a limited harmonic wave, that it is represented by

$$\cos \beta\,(x - vt),$$

but that the wave starts suddenly, continues for N complete wave-lengths and then stops suddenly. To find the effect of the prism on this train it is necessary first to reverse the ordinary procedure and, instead of analysing a pulse into waves, analyse the wave into pulses. Suppose the wave built up of a number of rectangular pulses, all of the same base length but having different heights. Then each of these pulses is dispersed by the prism into a system of wave trains; the wave trains for the different pulses differ among themselves in amplitude and phase. If we fix our attention on the point in the spectrum specified by α, in the limit when the pulses are made infinitely thin, the superposition of the wave trains gives at this point

$$\int \cos \beta v t_0 \cos \alpha\,(x - v\,\{t - t_0\})\,dt_0. \tag{11}$$

$= t_0$ gives the time at which the elementary pulse reaches the prism and $\cos \beta v t_0$ its height.

Let the resolving power at the wave-length in question be R, and let $2\pi \mathrm{R}/\alpha$, the length of the train into which a pulse is dispersed, be greater than $2\pi N/\beta$, the whole length of the incident train. Then for a length $2\pi(R/\alpha - N/\beta)$ all the wave trains overlap completely; on each side of this there is a length $2\pi N/\beta$, in which the overlapping takes place echelon-wise, at one end of which we have only a single train due to the first or last elementary pulse of the original train and, as we progress through which, the trains due to the other elementary pulses are gradually added, until at the other end we have all the trains due to the initial wave. For the middle region the integral (11) is to be taken over a range $2\pi N/(\beta v)$; for the end regions, as we move outwards from the middle, the range gradually diminishes from this value to zero.

Let us suppose that $\beta \neq \alpha$, then

$$\int \cos\beta vt_0 \cos\alpha(x-v\{t-t_0\})\,dt_0 = -\frac{1}{2(\alpha+\beta)v}\sin\{\alpha(x-vt)+vt_0(\alpha+\beta)\}$$

$$-\frac{1}{2(\alpha-\beta)v}\sin\{\alpha(x-vt)+vt_0(\alpha-\beta)\}.$$

If $\alpha = \beta$ the integral becomes

$$-\frac{1}{4\beta v}\sin\{\beta(x-vt)+2\beta vt_0\}+\frac{t_0}{2}\cos\beta(x-vt).$$

When the limits are substituted, it is obvious that it is only the second term of the second case that becomes important. Within the middle region it has a value

$$\frac{\pi N}{\beta v}\cos\beta(x-vt);$$

in the end regions the amplitude diminishes uniformly from $\pi N/(\beta v)$ to zero.

As was to be expected, therefore, the limited harmonic train has been refracted to one particular point in the spectrum. There it produces a train with $N+R$ waves, during the first N of which the amplitude steadily increases, for the next $R-N$ of which it is constant and for the last N of which it decreases. In the train there are no irregular changes of phase.

But when $\beta \neq \alpha$ the integral does not vanish. Consequently the other regions of the spectrum are not absolutely dark. With a bright line spectrum there must thus always be associated a faint background due simply to the fact that the harmonic trains producing the bright lines are limited. Theoretically it should be possible from a comparison of the intensities of the line and background to obtain a lower limit to the number of periods in the initial train.

§ 10. Let us suppose that instead of a limited harmonic train N equal pulses fall on the prism, one after the other, and that the time interval between the incidence of successive pulses is always the same and equal to t_0. Then the integral (11) gives place to a summation, namely,

$$\cos\alpha(x-vt)+\cos\alpha(x-v\{t-t_0\})+\ldots+\cos\alpha(x-v\{t-(\mathrm{N}-1)t_0\})$$
$$=\frac{\cos\{\alpha(x-vt)+\frac{1}{2}(\mathrm{N}-1)\alpha vt_0\}\sin\frac{1}{2}(\mathrm{N}\alpha vt_0)}{\sin\frac{1}{2}(\alpha vt_0)}.$$

The result may be regarded as a wave with a variable amplitude. The amplitude has a maximum equal to N, when $\alpha vt_0 = 2n\pi$, n being any integer. Now $\alpha = (2n\pi)/(vt_0)$ gives an infinite number of points in the spectrum. The train of pulses has thus given rise to something like a spectral series; the convergence frequency is, of course, away at the extreme end of the spectrum at wave-length zero. The lines of the series are quite as sharp as the maximum of a diffraction grating with N rulings, the expressions for the variation of the intensity being exactly the same in the two cases.

Modern investigation has shown spectra to be extraordinarily complicated. How far is it possible to introduce simplicity by ascribing different lines to the same common train of pulses, pulses delivered according to laws yet to be discovered, not necessarily at equal time intervals as in this case?

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*On Protection from Lightning and the Range of Protection afforded by Lightning Rods.**

By Sir JOSEPH LARMOR, F.R.S., and J. S. B. LARMOR, M.A.

(Received March 31,—Read May 7, 1914.)

The elasticity of the æther is perfect; and a perfect vacuum is infinitely strong, so that no electric force, however great, can produce disruptive discharge through it. The electric weakness of a rarefied gas is now known to be connected (J. S. Townsend, J. J. Thomson) with the long free paths of whatever ions may exist in it; their length allows time for the ions to attain abnormally high velocities, under the acceleration produced by the forces of an electric field, before they are deflected by collisions, and thus enables them to produce more ions in geometrical progression, by shattering impact as they come into collision with the molecules of the gas.

At ordinary densities, with their short free paths, it requires much stronger fields to get up the velocities requisite to induce such an explosively increasing

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